

## EQUATIONS OF RIEMANN SURFACES WITH AUTOMORPHISMS

DAVID SWINARSKI

**ABSTRACT.** We present an algorithm for computing equations of canonically embedded Riemann surfaces with automorphisms. A variant of this algorithm with many heuristic improvements is used to produce equations of Riemann surfaces  $X$  with large automorphism groups (that is,  $|\text{Aut}(X)| > 4(g_X - 1)$ ) for genus  $4 \leq g_X \leq 7$ . The main tools are the Eichler trace formula for the character of the action of  $\text{Aut}(X)$  on holomorphic differentials, algorithms for producing matrix generators of a representation of a finite group with a specified irreducible character, and Gröbner basis techniques for computing flattening stratifications.

Riemann surfaces (or algebraic curves) with automorphisms have been important objects of study in complex analysis, algebraic geometry, number theory, and theoretical physics for over a century, as their symmetries often permit us to do calculations that would otherwise be intractable.

Such Riemann surfaces are special in the sense that a general Riemann surface of genus  $g \geq 3$  has no nontrivial automorphisms. Moreover, the group of automorphisms of a Riemann surface of genus  $g \geq 2$  is finite.

Breuer and Conder performed computer searches that for each genus  $g$  list the Riemann surfaces of genus  $g$  with large automorphism groups (that is,  $|\text{Aut}(X)| > 4(g_X - 1)$ ). Specifically, they list sets of surface kernel generators (see Definition 1.2 below), which describe these Riemann surfaces as branched covers of  $\mathbb{P}^1$ . Breuer's list extends to genus  $g = 48$ , and Conder's list extends to genus  $g = 101$  [3, 6]. Even for small values of  $g$ , these lists are extremely large, as a surface  $X$  may appear several times for various subgroups of its full automorphism group. In [19], Magaard, Shaska, Shpectorov, and Völklein refined Breuer's list by determining which surface kernel generators correspond to the full automorphism group of the Riemann surface.

To my knowledge, at this time there is no general algorithm published in the literature for producing equations of these Riemann surfaces under any embedding from this data. Here, I present an algorithm to compute canonical equations of nonhyperelliptic Riemann surfaces with automorphisms. The main tools are the Eichler trace formula for the character of the action of  $\text{Aut}(X)$  on holomorphic differentials, algorithms for producing matrix generators of a representation of a finite group with a specified irreducible character, and Gröbner basis techniques for computing flattening stratifications. A variant of this algorithm with many heuristic improvements is used to produce equations of the nonhyperelliptic Riemann surfaces with genus  $4 \leq g_X \leq 7$  satisfying  $|\text{Aut}(X)| > 4(g_X - 1)$ .

Here is an outline of the paper. In Section 1, I describe the main algorithm. In Section 2, I describe several heuristics that simplify or speed up the main algorithm. In Section 3, I describe one example in detail, a genus 7 Riemann surface with 64 automorphisms. In Section 4, I give equations of selected canonically embedded Riemann surfaces with  $4 \leq g_X \leq 7$  along with matrix surface kernel generators.

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**Online material.** My webpage for this project is [28]. This page contains links to the latest version of my **Magma** code, files detailing the calculations for specific examples, and many equations that are omitted in the tables in Section 4.

In future work, Jennifer Paulhus and I plan to include much of the data described in this paper and on the website [28] in the L-Functions and Modular Forms Database at [lmfdb.org](http://lmfdb.org).

## 1. THE MAIN ALGORITHM

We begin by stating the main algorithm. Then, in the following subsections, we discuss each step in more detail, including precise definitions and references for terms and facts that are not commonly known.

### Algorithm 1.1.

INPUTS:

- (1) *A finite group  $G$ ;*
- (2) *an integer  $g \geq 2$ ;*
- (3) *a set of surface kernel generators  $(a_1, \dots, a_{g_0}; b_1, \dots, b_{g_0}; g_1, \dots, g_r)$  determining a family of nonhyperelliptic Riemann surfaces  $X$  of genus  $g$  with  $G \subset \text{Aut}(X)$*

OUTPUT: *A locally closed set  $B \subset \mathbb{A}^n$  and a family of smooth curves  $\mathcal{X} \subset \mathbb{P}^{g-1} \times B$  such that for each closed point  $b \in B$ , the fiber  $\mathcal{X}_b$  is a smooth genus  $g$  canonically embedded curve with  $G \subset \text{Aut}(\mathcal{X}_b)$ .*

Step 1. *Compute the conjugacy classes and character table of  $G$ .*

Step 2. *Use the Eichler trace formula to compute the character of the action on differentials and on cubics in the canonical ideal.*

Step 3. *Obtain matrix generators for the action on holomorphic differentials.*

Step 4. *Use the projection formula to obtain candidate cubics.*

Step 5. *Compute a flattening stratification and select the locus yielding smooth algebraic curves with degree  $2g - 2$  and genus  $g$ .*

**1.1. Step 1: conjugacy classes and character table of  $G$ .** This step is purely for bookkeeping. It is customary to list the conjugacy classes of  $G$  in increasing order, and to list the rows in a character table by increasing degree. However, there is no canonical order to either the conjugacy classes or the irreducible characters. Given two different descriptions of a finite group  $G$ , modern software such as **Magma** may order the classes or the irreducible characters of  $G$  differently. Hence, we compute and fix these at the beginning of the calculation.

**1.2. Step 2: Counting fixed points and the Eichler trace formula.** Here we define surface kernel generators for the automorphism group of a Riemann surface. These generators determine the Riemann surface as a branched cover of  $\mathbb{P}^1$  and are used in a key formula (see Theorem 1.3 below) for counting the number of fixed points of an automorphism.

**Definition 1.2** (cf. [3] Theorem 3.2, Theorem 3.14). A *signature* is a list of integers  $(g_0; e_1, \dots, e_r)$  with  $g_0 \geq 0$ ,  $r \geq 0$ , and  $e_i \geq 2$ .

A set of *surface kernel generators* for a finite group  $G$  and signature  $(g_0; e_1, \dots, e_r)$  is a sequence of elements  $a_1, \dots, a_{g_0}, b_1, \dots, b_{g_0}, g_1, \dots, g_r \in G$  such that

- (1)  $\langle a_1, \dots, a_{g_0}, b_1, \dots, b_{g_0}, g_1, \dots, g_r \rangle = G$ ;
- (2)  $\text{Order}(g_i) = e_i$ ; and
- (3)  $\prod_{j=1}^{g_0} [a_j, b_j] \prod_{i=1}^r g_i = \text{Id}_G$ .

Surface kernel generators have many other names in other papers; they are called *ramification types* in [19] and *generating vectors* in [23].

As explained in [3, Section 3.11], surface kernel generators describe the quotient morphism  $X \rightarrow X/G$  as a branched cover. Here  $X$  is a Riemann surface of genus  $g$ ,  $G$  is a subgroup of  $\text{Aut}(X)$ , the quotient  $X/G$  has genus  $g_0$ , the quotient morphism branches over  $r$  points, and the integers  $e_i$  describe the ramification over the branch points.

In the sequel we will be primarily interested in large automorphism groups, that is,  $|\text{Aut}(X)| > 4(g_X - 1)$ . In this case, the Riemann-Hurwitz formula implies that  $g_0 = 0$  and  $3 \leq r \leq 4$ .

Surface kernel generators are used in the following formula for the number of fixed points of an automorphism:

**Theorem 1.3** ([3, Lemma 11.5]). *Let  $\sigma$  be an automorphism of order  $h > 1$  of a Riemann surface  $X$  of genus  $g \geq 2$ . Let  $(g_1, \dots, g_r)$  be part of a set of surface kernel generators for  $X$ , and let  $(m_1, \dots, m_r)$  be the orders of these elements. Let  $\text{Fix}_{X,u}(\sigma)$  be the set of fixed points of  $X$  where  $\sigma$  acts on a neighborhood of the fixed point by  $z \mapsto \exp(2\pi i u/h)z$ . Then*

$$|\text{Fix}_{X,u}(\sigma)| = |C_G(\sigma)| \sum_{\substack{g_i \text{ s.t.} \\ h|m_i \\ \sigma \sim g_i^{m_i u/h}}} \frac{1}{m_i}$$

Here  $C_G(\sigma)$  is the centralizer of  $\sigma$  in  $G$ , and  $\sim$  denotes conjugacy.

Next we recall the Eichler Trace Formula. For a Riemann surface  $X$ , let  $\Omega_X$  be the holomorphic cotangent bundle, and let  $\omega_X = \bigwedge \Omega_X$  be the sheaf of holomorphic differentials. The Eichler Trace Formula gives the character of the action of  $\text{Aut}(X)$  on  $\Gamma(\omega_X^{\otimes d})$ .

**Theorem 1.4** (Eichler Trace Formula [10, Theorem V.2.9]). *Suppose  $g_X \geq 2$ , and let  $\sigma$  be a nontrivial automorphism of  $X$  of order  $h$ . Write  $\chi_d$  for the character of the representation of  $\text{Aut}(X)$  on  $\Gamma(\omega_X^{\otimes d})$ . Then*

$$\chi_d(\sigma) = \begin{cases} 1 + \sum_{\substack{1 \leq u < h \\ (u,h)=1}} |\text{Fix}_{X,u}(\sigma)| \frac{\zeta_h^u}{1 - \zeta_h^u} & \text{if } d = 1 \\ \sum_{\substack{1 \leq u < h \\ (u,h)=1}} |\text{Fix}_{X,u}(\sigma)| \frac{\zeta_h^{u(d \% h)}}{1 - \zeta_h^u} & \text{if } d \geq 2 \end{cases}$$

Together, the previous two results give a group-theoretic method for computing the character of the  $\text{Aut}(X)$  action on  $\Gamma(\omega_X^{\otimes d})$  starting from a set of surface kernel generators.

We can use the character of  $\text{Aut}(X)$  on  $\Gamma(\omega_X^{\otimes d})$  to obtain the character of  $\text{Aut}(X)$  on quadric and cubics in the canonical ideal as follows. Let  $S$  be the coordinate ring of  $\mathbb{P}^{g-1}$ , let  $I \subset S$  be the canonical ideal, and let  $S_d$  and  $I_d$  denote the degree  $d$  subspaces of  $S$  and  $I$ .

By Noether's Theorem, the sequence

$$0 \rightarrow I_d \rightarrow S_d \rightarrow \Gamma(\omega_X^{\otimes d}) \rightarrow 0$$

is exact for each  $d \geq 2$ , and by Petri's Theorem, the canonical ideal is generated either by quadrics or by quadrics and cubics. Thus, beginning with the character of the action on  $\Gamma(\omega_X) \cong S_1$ , we may compute the characters of the actions on  $S_2 = \text{Sym}^2 S_1$  and  $S_3 = \text{Sym}^3 S_1$  and  $\Gamma(\omega_X^{\otimes 2})$  and  $\Gamma(\omega_X^{\otimes 3})$ , and then obtain the characters of the actions on  $I_2$  and  $I_3$ .

**1.3. Step 3: matrix generators for a specified irreducible character.** From Step 2 we have the character of the action on  $\Gamma(\omega_X)$ . We seek matrix generators for this action. It suffices to find matrix generators for each irreducible  $G$ -module appearing in  $\Gamma(\omega_X)$ .

Given a finite group  $G$  and an irreducible character  $\chi$  of  $G$ , software such as **GAP** [12] and **Magma** contain commands for producing matrix generators of a representation  $V$  of  $G$  with character  $\chi$ . Finding efficient algorithms to produce matrix generators with good properties (for instance, sparse matrices, or matrices whose entries have small height, or matrices whose entries belong to a low degree extension of  $\mathbb{Q}$ ) is a subject of ongoing research [7, 8]. It seems that computer algebra systems implement several different algorithms that cover many special cases.

I do not know a reference for a general algorithm. Hence, I briefly present an algorithm that was suggested to me by Valery Alexeev and James McKernan. This algorithm is not expected to perform efficiently; it is included merely to establish that Step 3 in Algorithm 1.1 can be performed algorithmically.

#### Algorithm 1.5.

INPUTS:

- (1) a finite group  $G$  with generators  $g_1, \dots, g_r$ ;
- (2) an irreducible character  $\chi : G \rightarrow \mathbb{C}$  of degree  $n$ .

OUTPUT: matrices  $M_1, \dots, M_r \in \mathrm{GL}(n, \mathbb{C})$  such that the homomorphism  $g_i \mapsto M_i$  is a representation with character  $\chi$

- Step 1. Compute matrix generators for the regular representation  $V$  of  $G$ . These matrices are permutation matrices, and hence their entries are in  $\{0, 1\}$ .
- Step 2. Use the projection formula (see Theorem 1.6 below) to compute matrix generators  $\rho_W(g)$  for a representation  $W$  with character  $n\chi$ . Let  $K$  be the smallest field containing  $\{\chi(g) : g \in G\}$ . Note that  $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}[\zeta_{|G|}]$ . Then the matrix generators  $\rho_W(g)$  lie in  $\mathrm{GL}(n^2, K)$ .
- Step 3. Let  $x_1, \dots, x_{n^2}$  be indeterminates. Let  $M$  be the  $|G| \times n^2$  matrix over  $K$  whose rows are given by the vectors  $\rho_W(g) \cdot (x_1, \dots, x_{n^2})$ . Let  $X \subset \mathbb{P}_K^{n^2-1}$  be the determinantal variety  $\mathrm{rank} M \leq n$ . Since representations of finite groups are completely reducible in characteristic zero, the representation  $W$  is isomorphic over  $K$  to the direct sum  $V_\chi^{\oplus n}$ , and therefore  $X(K)$  is non-empty.
- Step 4. Intersect  $X$  with generic hyperplanes with coefficients in  $K$  to obtain a zero-dimensional variety  $Y$ .
- Step 5. If necessary, pass to a finite field extension  $L$  of  $K$  to obtain a reduced closed point  $y \in Y(L)$ .
- Step 6. The point  $y$  (regarded as a vector in  $W \otimes L$ ) generates the desired representation.

An example where this algorithm is used to produce matrix generators for the degree two irreducible representation of the symmetric group  $S_3$  is available at my webpage [28].

Finally, we note that in [27], Streit describes a method for producing matrix generators for the action of  $\mathrm{Aut}(X)$  on  $\Gamma(\omega_X)$  for some Belyi curves.

**1.4. Step 4: the projection formula.** Recall the projection formula for representations of finite groups. (See for instance [11] formula (2.31)).

**Theorem 1.6** (Projection formula). *Let  $V$  be a finite-dimensional representation of a finite group  $G$  over  $\mathbb{C}$ . Let  $V_1, \dots, V_k$  be the irreducible representations of  $G$ , let  $\chi_i$  be their characters, and let  $V \cong \bigoplus_{i=1}^k V_i^{\oplus m_i}$ . Let  $\pi : V \rightarrow V_i^{\oplus m_i}$  be the projection onto the  $i^{\mathrm{th}}$  isotypical component of  $V$ . Then*

$$\pi_i = \frac{\dim(V_i)}{|G|} \sum_{g \in G} \overline{\chi_i(g)} g.$$

From Step 3, we have matrix generators for the  $G$  action on  $\Gamma(\omega_X) = S_1$ . Thus, we can compute matrix generators for the actions on  $S_2$  and  $S_3$ , and use the projection formula to compute the isotypical subspace  $S_{d,p}$  of degree  $d$  polynomials on which  $G$  acts with character  $\chi_p$ . In some a few examples, we have  $I_{d,p} = S_{d,p}$ , but more commonly, we have strict containment  $I_{d,p} \subset S_{d,p}$ . In this case we write elements of  $I_{d,p}$  as generic linear combinations of the basis elements of  $S_{d,p}$  and then seek coefficients that yield a smooth algebraic curve with the correct degree and genus.

The coefficients used to form these generic linear combinations form the base space  $\mathbb{A}^n$  of the family  $\mathcal{X}$  produced by the main algorithm.

### 1.5. Step 5: Flattening stratifications.

**Theorem 1.7** ([21, Lecture 8]). *Let  $f : X \rightarrow S$  be a projective morphism with  $S$  a reduced Noetherian scheme. Then there exist locally closed subsets  $S_1, \dots, S_n$  such that  $S = \sqcup_{i=1}^n S_i$  and  $f|_{f^{-1}(S_i)}$  is flat.*

The stratification  $S = \sqcup_{i=1}^n S_i$  is called a flattening stratification for the map  $f$ . Since  $S$  is reduced, flatness implies that over each stratum, the Hilbert polynomial of the fibers is constant. We find the stratum with Hilbert polynomial  $P(t) = (2g - 2)t - g + 1$ , then intersect this stratum with the locus where the fibers are smooth. This completes the algorithm.

Flattening stratifications have been an important tool in theoretical algebraic geometry for over 50 years. There exist Gröbner basis techniques for computing flattening stratifications; in the computational literature, these are typically called *comprehensive* or *parametric Gröbner bases*, or *Gröbner systems*. The foundational work on this problem was begun by Weispfenning, and many authors, including Manubens and Montes, Suzuki and Sato, Nabeshima, and Kapur, Sun, and Wang, have made important improvements on the original algorithm [14, 22, 29].

The size of a Gröbner basis can grow very quickly with the number of variables and generators of an ideal, and unfortunately, even the most recent software cannot compute flattening stratifications for the examples we consider. Thus, in section 2.2 below, we discuss a strategy for circumventing this obstacle.

## 2. HEURISTIC IMPROVEMENTS

Many steps of Algorithm 1.1 can be run using a computer algebra system, but even for modest examples, the flattening stratification required in the final step is intractable. Therefore we discuss various heuristics that can be employed to speed the computation.

**2.1. Tests for gonality and reduction to quadrics.** Given a set of surface kernel generators, it is useful to discover as early as possible whether the corresponding Riemann surface is hyperelliptic, trigonal, a plane quintic, or none of these. We discuss these properties in turn.

*Hyperelliptic Riemann surfaces.* Algorithm 1.1 supposes that one begins with surface kernel generators corresponding to a nonhyperelliptic curve. However, we can easily test for hyperellipticity if this property is not known in advance. A Riemann surface  $X$  is hyperelliptic if and only if  $\text{Aut}(X)$  contains a central involution with  $2g_X + 2$  fixed points. Thus, given a set of surface kernel generators, we can search for a central involution and count its fixed points using Theorem 1.3 (or even better, using [3, Lemma 10.4]).

In [25], Shaska gives equations of the form  $y^2 = f(x)$  for hyperelliptic curves with automorphisms. Additionally, we can use the algorithm described in [26] to get the equations of  $C$  under a linear series such as the transcanonical embedding or bicanonical embedding.

So suppose the Riemann surface is not hyperelliptic. By Petri's Theorem, the canonical ideal is generated by quadrics if  $X$  is not hyperelliptic, not trigonal, and not a plane quintic. Thus, ruling out these possibilities allows us to work with quadrics instead of cubics, which significantly speeds up the algorithm. This leads us to consider trigonal Riemann surfaces and plane quintics.

*Trigonal Riemann surfaces.* Trigonal Riemann surfaces may be divided into two types: cyclic trigonal and general trigonal [5]. Cyclic trigonal curves can be detected by searching for degree three elements fixing  $g + 2$  points. Their automorphism groups have been classified [4, Theorem 2.1], and one may hope for a paper treating equations of cyclic trigonal Riemann surfaces as the paper [25] treats equations of hyperelliptic Riemann surfaces.

Less is known about general trigonal curves. We have Arakawa's bounds [2, Remark 5] and a few additional necessary conditions [5, Prop. 4 and Lemma 5]. We will not say more about general trigonal Riemann surfaces here because after studying the Riemann surfaces with large automorphism groups with genus  $4 \leq g \leq 7$ , we learn *a posteriori* that very few of them are general trigonal.

*Plane quintics.* Plane quintics only occur in genus 6, and the canonical model of a plane quintic lies on the Veronese surface in  $\mathbb{P}^5$ . Thus, we have a necessary condition:  $X$  is a plane quintic only if  $\Gamma(\omega_X) \cong \text{Sym}^2 V$  for some (possibly reducible) three-dimensional representation  $V$  of  $G$ . In practice, it is generally quite fast to discover whether a nonhyperelliptic non-cyclic trigonal genus 6 Riemann surface is a plane quintic.

**2.2. Partial flattening stratifications.** In this section we use several notions from the theory of Gröbner bases. We will not recall all the definitions here, and instead refer to [9, Chapter 15] for the details.

The algorithms for comprehensive Gröbner bases described in Section 1.5 all begin with the same observation. Let  $S = K[x_0, \dots, x_m]$  be a polynomial ring over a field. Let  $\preceq$  be a multiplicative term order on  $S$ . Then a theorem of Macaulay states that the Hilbert function of  $I$  is the same as the Hilbert function of its initial ideal with respect to this term order (see [9, Theorem 15.26]).

Therefore, whenever two ideals in  $S$  have Gröbner bases with the same leading monomials with respect to some term order, they will have the same initial ideal for that term order, hence they must have the same Hilbert function and Hilbert polynomial, and therefore they will lie in the same stratum of a flattening stratification. To reach a different stratum of the flattening stratification, it is necessary to alter the leading terms of the Gröbner basis — for instance, by restricting to the locus where that coefficient vanishes.

Here is a brief example to illustrate this idea. Let  $\mathbb{A}^2$  have coordinates  $c_1, c_2$ , and let  $\mathbb{P}^3$  have coordinates  $x_0, x_1, x_2, x_3$ . The ideal

$$I = \langle c_1 x_0 x_2 - c_2 x_1^2, c_1 x_0 x_3 - c_2 x_1 x_2, c_1 x_1 x_3 - c_2 x_2^2 \rangle$$

defines a 2-parameter family of subschemes of  $\mathbb{P}^3$ . A Gröbner basis for  $I$  in  $\mathbb{C}[c_1, c_2][x_0, x_1, x_2, x_3]$  with respect to the lexicographic term order is

$$\begin{aligned} & c_1x_0x_2 - c_2x_1^2, c_1x_0x_3 - c_2x_1x_2, c_1x_1x_3 - c_2x_2^2, \\ & (c_1c_2 - c_2^2)x_1x_2^2, c_2x_1^2x_3 - c_2x_1x_2^2, (c_1c_2 - c_2^2)x_1^2x_2, c_2x_0x_2^2 - c_2x_1^2x_2, \\ & (c_1c_2^2 - c_2^3)x_1^4, (c_1c_2^2 - c_2^3)x_2^4, c_2^2x_1x_2^2x_3 - c_2^2x_2^4, c_2^2x_0x_1^2x_2 - c_2^2x_1^4. \end{aligned}$$

Over the locus where  $c_1$ ,  $c_2$ , and  $c_1 - c_2$  are invertible, the initial ideal is  $\langle x_0x_2, x_0x_3, x_1x_3, x_1x_2^2, x_1^2x_2, x_1^4, x_2^4 \rangle$  with Hilbert polynomial  $P(t) = 8$ . On the other hand, when  $c_1 = 0$ , or  $c_2 = 0$ , or  $c_1 - c_2 = 0$ , we get a different initial ideal and Hilbert polynomial. For example, the locus  $c_1 = c_2 \neq 0$  yields the twisted cubic with  $P(t) = 3t + 1$ .

Note that to discover this locus, it is not necessary to compute the entire Gröbner basis; it would suffice for instance to compute the S-pair reduction for the first two generators, which yields  $(c_1c_2 - c_2^2)x_1x_2^2$ .

Modern software packages by Nabeshima, Montes, and Kapur, Sun, and Wang can completely analyze this example. However, these packages did not yield answers on the problems that arose in this work. Therefore, I used the strategy outlined above. I partially computed a Gröbner basis in `Macaulay2` [17], and set some coefficients to zero. Remarkably, this was sufficient to obtain the equations of the genus  $4 \leq g \leq 7$  Riemann surfaces with large automorphism groups. Some of the families analyzed in this manner had as many as six coefficients  $c_1, \dots, c_6$ .

### 3. EXAMPLE: A GENUS 7 RIEMANN SURFACE WITH 64 AUTOMORPHISMS

Magaard, Shaska, Shpectorov, and Völklein's tables show that there exists a smooth, compact genus 7 Riemann surface with automorphism group  $G$  given by the group labeled (64, 41) in the GAP library of small finite groups. It has  $X/G \cong \mathbb{P}^1$ . The quotient morphism is branched over 3 points of  $\mathbb{P}^1$ , and the ramification indices over these points are 2, 4, and 16.

A naive search for a set of surface kernel generators in this group yields elements  $g_1$  and  $g_2$  with orders 2 and 4 such that  $(g_1g_2)^{-1}$  has order 16. There are four relations among these generators:

$$g_1^2, g_2^4, (g_2^{-1}g_1)^2g_2^2g_1g_2g_1g_2^{-1}, (g_2g_1)^2g_2^{-1}(g_1g_2)^2g_1g_2^{-1}(g_1g_2)^2g_1$$

*Step 1.* We use `Magma` to compute the conjugacy classes and character table of  $G$ . There are 16 conjugacy classes. For convenience, write  $g_3 = g_1^{-1}g_2^{-1}g_1g_2$ . Then a list of representatives of the conjugacy classes is

$$\begin{aligned} & \text{Id}, g_3^4, g_2^2, g_1, g_3^2, g_2^2g_3^2, g_2^3, g_2, \\ & g_1g_2^2, g_3g_2^2g_3^4, g_3g_2^2, g_3, g_2g_1, g_2g_1g_2^2g_3^2, g_2g_1g_3^2, g_2g_1g_2^2 \end{aligned}$$

Next we compute the character table. The irreducible characters are given below by their values on the sixteen conjugacy classes.

$$\begin{aligned} \chi_1 &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \\ \chi_2 &= (1, 1, 1, -1, 1, 1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1) \\ \chi_3 &= (1, 1, 1, 1, 1, 1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1) \\ \chi_4 &= (1, 1, 1, -1, 1, 1, 1, 1, -1, 1, 1, 1, -1, -1, -1, -1) \\ \chi_5 &= (1, 1, -1, 1, 1, -1, -i, i, -1, -1, -1, 1, i, -i, i, -i) \\ \chi_6 &= (1, 1, -1, 1, 1, -1, i, -i, -1, -1, -1, 1, -i, i, -i, i) \\ \chi_7 &= (1, 1, -1, -1, 1, -1, i, -i, 1, -1, -1, 1, i, -i, i, -i) \\ \chi_8 &= (1, 1, -1, -1, 1, -1, -i, i, 1, -1, -1, 1, -i, i, -i, i) \end{aligned}$$

$$\begin{aligned}
\chi_9 &= (2, 2, -2, 0, 2, -2, 0, 0, 0, 2, 2, -2, 0, 0, 0, 0) \\
\chi_{10} &= (2, 2, 2, 0, 2, 2, 0, 0, 0, -2, -2, -2, 0, 0, 0, 0) \\
\chi_{11} &= (2, 2, 2, 0, -2, -2, 0, 0, 0, 0, 0, -\sqrt{2}, \sqrt{2}, \sqrt{2}, -\sqrt{2}) \\
\chi_{12} &= (2, 2, -2, 0, -2, 2, 0, 0, 0, 0, 0, \sqrt{2}i, \sqrt{2}i, -\sqrt{2}i, -\sqrt{2}i) \\
\chi_{13} &= (2, 2, 2, 0, -2, -2, 0, 0, 0, 0, 0, \sqrt{2}, -\sqrt{2}, -\sqrt{2}, \sqrt{2}) \\
\chi_{14} &= (2, 2, -2, 0, -2, 2, 0, 0, 0, 0, 0, -\sqrt{2}i, -\sqrt{2}i, \sqrt{2}i, \sqrt{2}i) \\
\chi_{15} &= (4, -4, 0, 0, 0, 0, 0, 0, 0, -\sqrt{8}i, \sqrt{8}i, 0, 0, 0, 0, 0) \\
\chi_{16} &= (4, -4, 0, 0, 0, 0, 0, 0, 0, \sqrt{8}i, -\sqrt{8}i, 0, 0, 0, 0, 0)
\end{aligned}$$

*Step 2.* Let  $V_i$  be the irreducible  $G$ -module with character  $\chi_i$  given by the table above. For any  $G$ -module  $V$ , let  $V \cong \bigoplus_{i=1}^r V_i^{\oplus m_i}$  be its decomposition into irreducible  $G$ -modules.

We use the Eichler trace formula in **Magma** to compute these multiplicities  $m_i$  for several relevant  $G$ -modules. Let  $S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5, x_6]$ , and let  $S_d$  denote polynomials of degree  $d$ . Let  $I_d$  be the kernel defined by

$$0 \rightarrow I_d \rightarrow S_d \rightarrow \Gamma(\omega_X^{\otimes d}) \rightarrow 0.$$

Then we have

$$\begin{aligned}
S_1 &\cong \Gamma(\omega_X) \cong V_8 \oplus V_{14} \oplus V_{15} \\
I_2 &\cong V_3 \oplus V_5 \oplus V_{10} \oplus V_{11} \oplus V_{16} \\
S_2 &\cong V_3^{\oplus 2} \oplus V_5 \oplus V_6 \oplus V_{10}^{\oplus 2} \oplus V_{11}^{\oplus 2} \oplus V_{13} \oplus V_{14} \oplus V_{15} \oplus V_{16}^{\oplus 2} \\
\Gamma(\omega_X^{\otimes 2}) &\cong V_3 \oplus V_6 \oplus V_{10} \oplus V_{11} \oplus V_{13} \oplus V_{14} \oplus V_{15} \oplus V_{16}
\end{aligned}$$

We use **GAP** to obtain matrix representatives of a  $G$  action with character equal to the character of the  $G$  action on  $S_1$ . Such a representation is obtained by mapping the generators  $g_1$  and  $g_2$  to the matrices below.

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \zeta_8^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\zeta_8 & 0 & 0 & 0 & 0 \\ 0 & -\zeta_8^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -\zeta_8 & 0 \\ 0 & 0 & 0 & 0 & -\zeta_8^3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The decomposition of  $S_1$  as a sum of three irreducible  $G$ -modules gives rise to the block diagonal form of these matrices.

*Step 4.* We use the projection formula in **Magma** to decompose the  $G$ -module of quadrics  $S_2$  into its isotypical components. When an isotypical component has multiplicity greater than 1, we (noncanonically) choose ordered bases so that the  $G$  action is given by the same matrices on each ordered basis.

$$\begin{aligned}
S_{2,3} &\cong V_3^{\oplus 2} = \langle x_0^2 \rangle \oplus \langle x_1 x_2 \rangle \\
S_{2,5} &\cong V_5 = \langle x_3 x_4 - \zeta_8 x_5 x_6 \rangle \\
S_{2,10} &\cong V_{10}^{\oplus 2} = \langle x_1^2, x_2^2 \rangle \oplus \langle x_3 x_6 + i x_4 x_5, i x_3 x_6 + x_4 x_5 \rangle \\
S_{2,11} &\cong V_{11}^{\oplus 2} = \langle x_0 x_1, x_0 x_2 \rangle \oplus \langle x_3^2 + \zeta_8^3 x_5^2, -x_4^2 - \zeta_8^3 x_6^2 \rangle \\
S_{2,16} &\cong V_{16}^{\oplus 2} = \langle x_0 x_3, x_0 x_4, x_0 x_5, x_0 x_6 \rangle \oplus \langle -\zeta_8 x_2 x_6, \zeta_8 x_1 x_5, -x_2 x_4, x_1 x_3 \rangle
\end{aligned}$$

The first isotypical subspace yields a polynomial of the form  $c_1 x_0^2 + c_2 x_1 x_2$ . We may assume that  $c_1$  and  $c_2$  are nonzero, scale  $x_0$  to make  $c_1 = c_2$ , and then divide by  $c_1$  to obtain the polynomial  $x_0^2 + x_1 x_2$ .

The second isotypical subspace yields the polynomial  $x_3 x_4 - \zeta_8 x_5 x_6$ .

The third isotypical subspace yields polynomials of the form  $c_3 x_0 x_1 + c_4 (x_3^2 + \zeta_8^3 x_5^2)$  and  $c_3 x_0 x_2 + c_4 (-x_4^2 - \zeta_8^3 x_6^2)$ . We assume that  $c_3$  and  $c_4$  are nonzero, scale  $x_1, x_2$  to make  $c_3 = c_4$ , then divide by  $c_3$  and  $c_4$ .

In the remaining isotypical subspaces no further scaling is possible, and hence we are left with two undetermined coefficients  $c_6$  and  $c_8$ .

Thus, a Riemann surface in this family has an ideal of the form

$$\begin{aligned}
& x_0^2 + x_1x_2 \\
& x_3x_4 - \zeta_8x_5x_6 \\
& x_1^2 + x_3x_6 + ix_4x_5 \\
& x_2^2 + ix_3x_6 + x_4x_5 \\
& x_0x_1 + c_6(x_3^2 + \zeta_8^3x_5^2) \\
& x_0x_2 + c_6(-x_4^2 - \zeta_8^3x_6^2) \\
& x_0x_3 + c_8(-\zeta_8x_2x_6) \\
& x_0x_4 + c_8(\zeta_8x_1x_5) \\
& x_0x_5 + c_8(-x_2x_4) \\
& x_0x_6 + c_8(x_1x_3)
\end{aligned}$$

*Step 5.* To find values of the coefficients  $c_6, c_8$  that yield a smooth curve, we partially compute a flattening stratification. Begin Buchberger's algorithm. We compute the S-pair reductions between the generators and find that

$$\begin{aligned}
S(f_1, f_6) &\rightarrow (c_6c_8 + \zeta_8^{-1})x_1x_4x_5 + \cdots \\
S(f_1, f_9) &\rightarrow (c_8^2 - \zeta_8^{-1})x_1x_2x_5 + \cdots
\end{aligned}$$

Therefore, in Buchberger's algorithm, these polynomials will be added to the Gröbner basis. This suggests that we study the locus given by the equations  $c_8^2 - \zeta_8^{-1} = 0$  and  $c_6c_8 + \zeta_8^{-1} = 0$  as an interesting stratum in the flattening stratification.

We check in **Magma** that the values  $c_6 = \zeta_{16}^7$  and  $c_8 = \zeta_{16}^{-1}$  yield a smooth genus 7 curve in  $\mathbb{P}^6$  with the desired automorphism group.

From these equations, we can compute the Betti table of this ideal:

1					
	10	16	3		
		3	16	10	
					1

Schreyer has classified Betti tables of genus 7 canonical curves in [24]. This Betti table implies that the curve is tetragonal (there exists a degree 4 morphism  $C \rightarrow \mathbb{P}^1$ ) but not trigonal or hyperelliptic, and it has no degree 6 morphism  $C \rightarrow \mathbb{P}^2$ .

#### 4. RESULTS

This project had two goals. The first goal was to establish that the heuristics described in Section 2 allow us to run a variant of the main algorithm to completion for genus  $4 \leq g \leq 7$  Riemann surfaces with large automorphism groups. To this end, for each Riemann surface from Table 4 of [19], the website [28] contains a link to a calculation where a variant of the main algorithm is used to produce equations.

The surface kernel generators needed to begin the algorithm were generally obtained by a naive search through the triples or quadruples in the groups listed in Table 4 of [19]. However, my **Magma** code also includes functions allowing the user to input surface kernel generators from any type of group, or to put in matrix surface kernel generators with the desired representation on  $\Gamma(\omega_X)$ . We note that Breuer's data has been recently extended and republished by Paulhus [23], and Conder's data is available online [6], so these sources could be used instead.

The equations obtained depend strongly on the matrix generators of the representation  $\text{Aut}(X)$  on  $\Gamma(\omega_X)$ . I generally obtained these matrices from **Magma**, **GAP**, the papers [15, 16], or [3, Appendix B], and thus had little control over this step. Indeed, in a few cases, the resulting equations are almost comically bad; for an example of this, compare my equations at [28] for the genus 7 curve with 504 automorphisms to Macbeath's equations for this curve. Given this, it is perhaps surprising that in most cases, the algorithm produces reasonable equations (i.e., polynomials supported on a small number of monomials with small coefficients).

The second goal of this project was to create a reference that would contain the most useful information about the equations and automorphisms of these curves. Thus, in this section, I print the best equations and automorphisms that I know, whether these were found in the literature or by the main algorithm. Many of the equations for genus  $4 \leq g \leq 6$  are classical, and references are given whenever possible. However, the



matrix surface kernel generators are not always equally easy to find. The equations for the genus 7 curves are almost all new, as are most of the 1-parameter families on the website [28].

**4.1. Description of the tables.** In the following tables I give equations for the Riemann surfaces of genus  $4 \leq g \leq 7$  with large automorphism groups that are unique in moduli ( $\delta = 0$  in the notation of Table 4 of [19]). The 1-parameter families ( $\delta = 1$ ) are not printed here but can be found on the website [28]. I order the examples the same way they appear in [19].

For hyperelliptic Riemann surfaces, I give an equation of the form  $y^2 = f(x)$ . Many of these are classically known, and all of them can be found in [25].

For plane quintics in genus 6, we give the plane quintic and surface kernel generators in  $\text{GL}(3, \mathbb{C})$ . The canonical ideal and  $G$  action can be easily computed from this data.

For nonhyperelliptic curves that are not plane quintics, we print equations of the canonical ideals and surface kernel generators as elements of  $\text{GL}(g, \mathbb{C})$ . Whenever such a matrix  $M \in \text{GL}(g, \mathbb{C})$  is sufficiently sparse, I frequently write the product  $M[x_0, \dots, x_{g-1}]^t$  to save space.

For the cyclic trigonal equations, I also print a cyclic trigonal equation, that is, one of the form  $y^3 = \prod_{i=1}^{d_1} (x - \alpha_i) \prod_{i=1}^{d_2} (x - \beta_i)$ , following the notation of [1, Section 2.5] (where cyclic trigonal curves are also called *trielliptic*).

Throughout the tables below, canonical ideals are shown in the polynomial ring  $\mathbb{C}[x_0, \dots, x_{g-1}]$ . The symbol  $\zeta_n$  denotes  $e^{2\pi i/n}$ , and we write  $i$  for  $\zeta_4$ .

**4.2. Genus 4.** In genus 4, every Riemann surface is either hyperelliptic or trigonal. Of the nine entries in Table 4 of [19], four are hyperelliptic, four are cyclic trigonal, and one is general trigonal.

Note: the Riemann surface with automorphism group  $(120, 34) = S_5$  is known as Bring's curve. Its best-known embedding is in  $\mathbb{P}^5$ , with equations  $\sum_{i=0}^4 x_i$ ,  $\sum_{i=0}^4 x_i^2$ ,  $\sum_{i=0}^4 x_i^3$ .

Genus 4, Locus 1: Group  $(120, 34) = S_5$ , signature  $(2, 4, 5)$ , general trigonal

$$\begin{aligned} \text{Ideal: } & x_0^2 + x_0x_1 + x_1^2 - x_1x_2 + x_2^2 - x_2x_3 + x_3^2, \\ & x_0^2x_1 + x_0x_1^2 + x_1^2x_2 - x_1x_2^2 + x_2^2x_3 - x_2x_3^2 \\ \text{Maps: } & (x_0, x_1, x_2, x_3) \mapsto (-x_0, -x_1, -x_2, -x_2 + x_3), \\ & (x_0, x_1, x_2, x_3) \mapsto (x_0 + x_1, -x_0 - x_2, -x_0 - x_3, -x_3) \end{aligned}$$

Genus 4, Locus 2: Group  $(72, 42)$ , signature  $(2, 3, 12)$ , cyclic trigonal

Trigonal equation:  $y^3 = x(x^4 - 1)$

Ideal:  $x_1x_3 - x_2^2$ ,

$$x_0^3 - x_1^2x_2 + x_2x_3^2$$

$$\text{Maps: } \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -\zeta_6 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\zeta_{12} & \frac{1}{2}\zeta_3 & \frac{1}{2}\zeta_{12} \\ 0 & \zeta_{12} & 0 & \zeta_{12} \\ 0 & -\frac{1}{2}\zeta_{12} & -\frac{1}{2}\zeta_3 & \frac{1}{2}\zeta_{12} \end{bmatrix},$$

Genus 4, Locus 3: Group  $(72, 40)$ , signature  $(2, 4, 6)$ , cyclic trigonal

Trigonal equation:  $y^3 = (x^3 - 1)^2(x^3 + 1)$

Ideal:  $x_0x_3 - x_1x_2$ ,

$$x_1^3 - x_0^3 - x_3^3 - x_2^3$$

$$\begin{aligned} \text{Maps: } & (x_0, x_1, x_2, x_3) \mapsto (-x_0, x_2, x_1, -x_3) \\ & (x_0, x_1, x_2, x_3) \mapsto (-x_2, \zeta_6^2x_0, \zeta_6x_3, x_1) \end{aligned}$$

Genus 4, Locus 4: Group  $(40, 8)$ , signature  $(2, 4, 10)$ , hyperelliptic

$$y^2 = x^{10} - 1$$

Genus 4, Locus 5: Group  $(36, 12)$ , signature  $(2, 6, 6)$ , cyclic trigonal

Trigonal equation:  $y^3 = (x^3 - 1)(x^3 + 1)$

Ideal:  $x_1x_3 - x_2^2$ ,

$$x_0^3 - x_3^3 + x_1^3$$

$$\begin{aligned} \text{Maps: } & (x_0, x_1, x_2, x_3) \mapsto (-x_0, \zeta_3x_3, -x_2, -\zeta_6x_1) \\ & (x_0, x_1, x_2, x_3) \mapsto (\zeta_3x_0, -\zeta_3x_3, \zeta_6x_2, -x_1) \end{aligned}$$

Genus 4, Locus 6: Group (32,19), signature (2,4,16), hyperelliptic  
 $y^2 = x^9 - x$

Genus 4, Locus 7: Group (24,3), signature (3,4,6), hyperelliptic  
 $y^2 = x(x^4 - 1)(x^4 + 2i\sqrt{3} + 1)$

Genus 4, Locus 8: Group (18,2), signature (2,9,18), hyperelliptic  
 $y^2 = x^9 - 1$

Genus 4, Locus 9: Group (15,1), signature (3,5,15), cyclic trigonal  
 Trigonal equation:  $y^3 = x^5 - 1$

Ideal:  $x_1x_3 - x_2^2,$   
 $x_0^3 - x_1^2x_2 + x_3^3$

Maps:  $(x_0, x_1, x_2, x_3) \mapsto (\zeta_3^2x_0, \zeta_3x_1, \zeta_3x_2, \zeta_3x_3)$   
 $(x_0, x_1, x_2, x_3) \mapsto (\zeta_5x_0, \zeta_5^3x_1, \zeta_5^2x_2, \zeta_5x_3)$

**4.3. Genus 5.** Of the ten entries in Table 4 of [19], five are hyperelliptic, and one is cyclic trigonal. The remaining four are general, hence their canonical models are complete intersections of three quadrics.

Genus 5, Locus 1: Group (192,181), signature (2,3,8)

Ideal: Wiman, [30]:

$$\begin{aligned} &x_0^2 + x_3^2 + x_4^2, \\ &x_1^2 + x_3^2 - x_4^2 \\ &x_2^2 + x_3x_4 \end{aligned}$$

$$\text{Maps: } \begin{bmatrix} 0 & 0 & \frac{1}{2}(i+1) & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1-i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\begin{bmatrix} 0 & \zeta_8^{-1} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -1-i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(i-1) & -\frac{1}{2}(i+1) \\ 0 & 0 & 0 & -\frac{1}{2}(i-1) & -\frac{1}{2}(i+1) \end{bmatrix}$$

Genus 5, Locus 2: Group (160,234), signature (2,4,5)

Ideal: Wiman, [30]:

$$\begin{aligned} &x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ &x_0^2 + \zeta_5x_1^2 + \zeta_5^2x_2^2 + \zeta_5^3x_3^2 + \zeta_5^4x_4^2, \\ &\zeta_5^4x_0^2 + \zeta_5^3x_1^2 + \zeta_5^2x_2^2 + \zeta_5x_3^2 + x_4^2 \end{aligned}$$

Maps:  $(x_0, x_1, x_2, x_3, x_4) \mapsto (-x_3, x_2, x_1, -x_0, -x_4),$   
 $(x_0, x_1, x_2, x_3, x_4) \mapsto (-x_0, x_4, -x_3, x_2, -x_1)$

Genus 5, Locus 3: Group (120,35), signature (2,3,10), hyperelliptic  
 $y^2 = x^{11} + 11x^6 - x$

Genus 5, Locus 4: Group (96,195), signature (2,4,6)

Ideal: Wiman, [30]:

$$\begin{aligned} &x_0^2 + x_3^2 + x_4^2, \\ &x_1^2 + \zeta_3x_3^2 + \zeta_3^2x_4^2, \\ &x_2^2 + \zeta_3^2x_3^2 + \zeta_3x_4^2 \end{aligned}$$

Maps:  $(x_0, x_1, x_2, x_3, x_4) \mapsto (-x_2, -x_1, -x_0, \zeta_3^2x_4, \zeta_3x_3),$   
 $(x_0, x_1, x_2, x_3, x_4) \mapsto (-x_0, x_2, -x_1, -x_4, x_3)$

Genus 5, Locus 5: Group (64,32), signature (2,4,8)

Ideal: Wiman, [30]:

$$\begin{aligned} x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ x_0^2 + ix_1^2 - x_2^2 - ix_3^2, \\ x_0^2 - x_1^2 + x_2^2 - x_3^2 \end{aligned}$$

Maps:  $(x_0, x_1, x_2, x_3, x_4) \mapsto (-x_0, x_1, -x_2, -x_3, -x_4),$   
 $(x_0, x_1, x_2, x_3, x_4) \mapsto (ix_1, -ix_2, ix_3, -ix_0, ix_4)$

Genus 5, Locus 6: Group (48,14), signature (2,4,12), hyperelliptic  
 $y^2 = x^{12} - 1$

Genus 5, Locus 7: Group (48,30), signature (3,4,4), hyperelliptic  
 $y^2 = x^{12} - 33x^8 - 33x^4 + 1$

Genus 5, Locus 8: Group (40,5), signature (2,4,20), hyperelliptic  
 $y^2 = x^{11} - x$

Genus 5, Locus 9: Group (30,2), signature (2,6,15), cyclic trigonal  
 Trigonal equation:  $y^3 = (x^5 - 1)x^2$

Ideal:  $x_0x_3 - x_1x_2, \quad x_0x_4 - x_1x_3, \quad x_2x_4 - x_3^2,$   
 $x_0^2x_1 - x_3x_4^2 + x_2^3,$   
 $x_0x_1^2 - x_4^3 + x_2^2x_3$

Maps:  $(x_0, x_1, x_2, x_3, x_4) \mapsto (\zeta_5x_1, \zeta_5^4x_0, -\zeta_5^2x_4, -x_3, -\zeta_5^3x_2),$   
 $(x_0, x_1, x_2, x_3, x_4) \mapsto (\zeta_{15}^{14}x_1, \zeta_{15}^{11}x_0, -\zeta_{15}^{13}x_4, -\zeta_{15}^{10}x_3, -\zeta_{15}^7x_2)$

Genus 5, Locus 10: Group (22,2), signature (2,11,22), hyperelliptic  
 $y^2 = x^{11} - 1$

**4.4. Genus 6.** Table 4 in [19] contains eleven entries for genus 6 Riemann surfaces with large automorphism groups and no moduli ( $\delta = 0$ ). Of these, four are hyperelliptic, three are cyclic trigonal, and three are plane quintics; only one is general.

For the plane quintics, we give the plane quintic equation in the variables  $y_0, y_1, y_2$ , and surface kernel generators acting on the plane. The canonical model of a plane quintic lies on the Veronese surface, and the multiples of the quintic by  $y_0, y_1, y_2$  may be encoded as cubics in  $x_0, \dots, x_5$ .

Genus 6, Locus 1: Group (150,5), signature (2,3,10), plane quintic

Plane quintic equation:  $y_0^5 + y_1^5 + y_2^5$

Maps:  $(y_0, y_1, y_2) \mapsto (-\zeta_5^3y_1, -\zeta_5^2y_0, -y_2),$   
 $(y_0, y_1, y_2) \mapsto (-\zeta_5^3y_1, -\zeta_5^2y_0, -y_2)$

Genus 6, Locus 2: Group (120,34) =  $S_5$ , signature (2,4,6)

Ideal: Inoue and Kato, [13]:

$$\begin{aligned} -x_0x_2 + x_1x_2 - x_0x_3 + x_1x_4, \\ -x_0x_1 + x_1x_2 - x_0x_3 + x_2x_5, \\ -x_0x_1 - x_0x_2 - 2x_0x_3 - x_3x_4 - x_3x_5, \\ -x_0x_1 - x_0x_2 - x_0x_3 - x_1x_4 - x_3x_4 - x_4x_5, \\ -x_0x_1 - x_0x_2 - x_0x_3 - x_2x_5 - x_3x_5 - x_4x_5, \\ 2(\sum_{i=1}^6 x_i^2) + x_0x_1 + x_0x_2 + x_1x_2 + 2x_1x_3 + 2x_2x_3 \\ + 2x_0x_4 + 2x_2x_4 + x_3x_4 + 2x_0x_5 + 2x_1x_5 + x_3x_5 + x_2x_5 \end{aligned}$$

Maps:  $\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix}$

Genus 6, Locus 3: Group (72,15), signature (2,4,9), cyclic trigonal

Trigonal equation:  $y^3 = (x^4 - 2\sqrt{3}ix^2 + 1)(x^4 + 2\sqrt{3}ix^2 + 1)^2$

Ideal:  $x_0x_2 - x_1^2, x_0x_4 - x_1x_3, x_0x_5 - x_1x_4,$   
 $x_1x_4 - x_2x_3, x_1x_5 - x_2x_4, x_3x_5 - x_4^2,$   
 $x_0^3 + (4\zeta_6 - 2)x_0^2x_2 + x_0x_2^2 + x_3^3 + (-4\zeta_6 + 2)x_3^2x_5 + x_3x_5^2$   
 $x_0^2x_1 + (4\zeta_6 - 2)x_0x_1x_2 + x_1x_2^2 + x_3^2x_4 + (-4\zeta_6 + 2)x_3x_4x_5 + x_4x_5^2,$   
 $x_0^2x_2 + (4\zeta_6 - 2)x_0x_2^2 + x_2^3 + x_3^2x_5 + (-4\zeta_6 + 2)x_3x_5^2 + x_5^3$

Maps: 
$$\begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2}\zeta_9 & \frac{1}{2}\zeta_{36}^{13} & \frac{1}{2}\zeta_9 \\ 0 & 0 & 0 & -\zeta_{36}^{13} & 0 & -\zeta_{36}^{13} \\ 0 & 0 & 0 & \frac{1}{2}\zeta_9 & \frac{1}{2}\zeta_{36}^{13} & -\frac{1}{2}\zeta_9 \\ \frac{1}{2}\zeta_{36}^{14} & \frac{1}{2}\zeta_{36}^5 & -\frac{1}{2}\zeta_{36}^{14} & 0 & 0 & 0 \\ -\zeta_{36}^5 & 0 & -\zeta_{36}^5 & 0 & 0 & 0 \\ -\frac{1}{2}\zeta_{36}^{14} & \frac{1}{2}\zeta_{36}^5 & \frac{1}{2}\zeta_{36}^{14} & 0 & 0 & 0 \end{bmatrix},$$

$(x_0, \dots, x_5) \mapsto (\zeta_{12}x_3, \zeta_3x_4, -\zeta_{12}x_5, \zeta_{12}^5x_0, \zeta_3^2x_1, -\zeta_{12}^5x_2)$

Genus 6, Locus 4: Group (56,7), signature (2,4,14), hyperelliptic  
 $y^2 = x^{14} - 1$

Genus 6, Locus 5: Group (48,6), signature (2,4,24), hyperelliptic  
 $y^2 = x^{13} - x$

Genus 6, Locus 6: Group (48,29), signature (2,6,8), hyperelliptic  
 $y^2 = x(x^4 - 1)(x^8 + 14x^4 + 1)$

Genus 6, Locus 7: Group (48,15), signature (2,6,8), cyclic trigonal

Trigonal equation:  $y^3 = (x^4 - 1)^2(x^4 + 1)$

Ideal:  $x_0x_2 - x_1^2, x_0x_4 - x_1x_3, x_0x_5 - x_1x_4,$   
 $x_1x_4 - x_2x_3, x_1x_5 - x_2x_4, x_3x_5 - x_4^2,$   
 $x_0x_1^2 - x_2^3 - x_3x_4^2 - x_5^3,$   
 $x_0^2x_1 - x_1x_2^2 - x_3^2x_4 - x_4x_5^2,$   
 $x_0^3 - x_1^2x_2 - x_3^3 - x_4^2x_5$

Maps:  $(x_0, \dots, x_5) \mapsto (\zeta_8^3x_5, -ix_4, \zeta_8x_3, -\zeta_8^3x_2, ix_1, -\zeta_8x_0),$   
 $(x_0, \dots, x_5) \mapsto (-\zeta_6x_2, \zeta_6x_1, -\zeta_6x_0, -\zeta_3x_5, \zeta_3x_4, -\zeta_3x_3)$

Genus 6, Locus 8: Group (39,1), signature (3,3,13), plane quintic

Plane quintic equation:  $y_0^4y_1 + y_1^4y_2 + y_2^4y_0$

Maps:  $(y_0, y_1, y_2) \mapsto (\zeta_{13}^4y_1, \zeta_{13}^{10}y_2, \zeta_{13}^{12}y_0),$   
 $(y_0, y_1, y_2) \mapsto (\zeta_{13}^8y_2, \zeta_{13}^7y_0, \zeta_{13}^{11}y_1)$

Genus 6, Locus 9: Group (30,1), signature (2,10,15), plane quintic

Plane quintic equation:  $y_0^5 + y_1^4y_2 + 2\zeta_5y_1^3y_2^2 + 2\zeta_5^2y_1^2y_2^3 + \zeta_5^3y_1y_2^4$

Maps:  $(y_0, y_1, y_2) \mapsto (y_0, \zeta_5y_2, \zeta_5^4y_1),$   
 $(y_0, y_1, y_2) \mapsto (\zeta_5^3y_0, -\zeta_5y_1 - \zeta_5^2y_2, \zeta_5)$

Genus 6, Locus 10: Group (26,2), signature (2,13,26), hyperelliptic  
 $y^2 = x^{13} - 1$

Genus 6, Locus 11: Group (21,2), signature (3,7,21), cyclic trigonal

Trigonal equation:  $y^3 = x^7 - 1$

Ideal:  $x_0x_3 - x_1x_2, x_0x_4 - x_1x_3, x_0x_5 - x_1x_4,$   
 $x_2x_4 - x_3^2, x_2x_5 - x_3x_4, x_3x_5 - x_4^2,$   
 $x_0^3 - x_3x_5^2 + x_2^3,$   
 $x_0^2x_1 - x_4x_5^2 + x_2^2x_3,$   
 $x_0x_1^2 - x_3^3 + x_2^2x_4$

Maps:  $(x_0, \dots, x_5) \mapsto (x_0, \zeta_7x_1, x_2, \zeta_7x_3, \zeta_7^2x_4, \zeta_7^3x_5),$   
 $(x_0, \dots, x_5) \mapsto (\zeta_3^2x_0, \zeta_3^2x_1, \zeta_3x_2, \zeta_3x_3, \zeta_3x_4, \zeta_3x_5)$

4.5. **Genus 7.** Of the thirteen entries in Table 4 of [19] for genus 7 curves, three are hyperelliptic and two are cyclic trigonal. After computing the canonical equations of the nonhyperelliptic Riemann surfaces, we can compute the Betti tables of these ideals and use the results of [24] to classify the curve as having a  $g_4^1$ ,  $g_6^2$ ,  $g_3^1$ , or none of these.

Genus 7, Locus 1: Group (504, 156), signature (2,3,7)

Ideal: Macbeath, [18]:

$$\begin{aligned} & x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2, \\ & x_0^2 + \zeta_7 x_1^2 + \zeta_7^2 x_2^2 + \zeta_7^3 x_3^2 + \zeta_7^4 x_4^2 + \zeta_7^5 x_5^2 + \zeta_7^6 x_6^2, \\ & x_0^2 + \zeta_7^{-1} x_1^2 + \zeta_7^{-2} x_2^2 + \zeta_7^{-3} x_3^2 + \zeta_7^{-4} x_4^2 + \zeta_7^{-5} x_5^2 + \zeta_7^{-6} x_6^2, \\ & (\zeta_7^{-3} - \zeta_7^3) x_0 x_6 - (\zeta_7^{-2} - \zeta_7^2) x_1 x_4 + (\zeta_7 - \zeta_7^{-1}) x_3 x_5, \\ & (\zeta_7^{-3} - \zeta_7^3) x_1 x_0 - (\zeta_7^{-2} - \zeta_7^2) x_2 x_5 + (\zeta_7 - \zeta_7^{-1}) x_4 x_6, \\ & (\zeta_7^{-3} - \zeta_7^3) x_2 x_1 - (\zeta_7^{-2} - \zeta_7^2) x_3 x_6 + (\zeta_7 - \zeta_7^{-1}) x_5 x_0, \\ & (\zeta_7^{-3} - \zeta_7^3) x_3 x_2 - (\zeta_7^{-2} - \zeta_7^2) x_4 x_0 + (\zeta_7 - \zeta_7^{-1}) x_6 x_1, \\ & (\zeta_7^{-3} - \zeta_7^3) x_4 x_3 - (\zeta_7^{-2} - \zeta_7^2) x_5 x_1 + (\zeta_7 - \zeta_7^{-1}) x_0 x_2, \\ & (\zeta_7^{-3} - \zeta_7^3) x_5 x_4 - (\zeta_7^{-2} - \zeta_7^2) x_6 x_2 + (\zeta_7 - \zeta_7^{-1}) x_1 x_3, \\ & (\zeta_7^{-3} - \zeta_7^3) x_6 x_5 - (\zeta_7^{-2} - \zeta_7^2) x_0 x_3 + (\zeta_7 - \zeta_7^{-1}) x_2 x_4 \end{aligned}$$

Maps:  $(x_0, \dots, x_6) \mapsto (x_0, -x_1, -x_2, -x_3, x_4, x_5, -x_6),$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

Genus 7, Locus 2: Group (144, 127), signature (2,3,12). Has  $g_6^2$

Ideal:  $x_0^2 + x_3 x_4 - \zeta_6 x_3 x_5 - \zeta_6 x_5 x_6,$   
 $2ix_1^2 + x_3 x_4 + \zeta_6 x_3 x_5 + 2x_4 x_6 - \zeta_6 x_5 x_6,$   
 $2ix_1 x_2 + (-2\zeta_6 + 1)x_3 x_4 + \zeta_6 x_3 x_5 - 2\zeta_6 x_4 x_6 + \zeta_6 x_5 x_6,$   
 $2ix_2^2 - x_3 x_4 + (-\zeta_6 + 2)x_3 x_5 + (2\zeta_6 - 2)x_4 x_6 + (-\zeta_6 + 2)x_5 x_6,$   
 $x_1 x_3 - \zeta_6 x_2 x_6 + \zeta_{12} x_4^2 + (\zeta_{12}^3 - 2\zeta_{12}) x_4 x_5 + \zeta_{12} x_5^2,$   
 $x_1 x_4 - \zeta_3 x_2 x_5 - x_3 x_6 - x_6^2,$   
 $x_1 x_5 - x_2 x_5 + x_3^2 + (-\zeta_6 + 2)x_3 x_6,$   
 $x_1 x_6 + x_2 x_6 - \zeta_{12} x_4^2 + \zeta_{12} x_4 x_5 - \zeta_{12} x_5^2,$   
 $x_2 x_3 - \zeta_6 x_2 x_6 + \zeta_{12} x_4^2,$   
 $x_2 x_4 + (-\zeta_6 - 1)x_2 x_5 + \zeta_6 x_3^2 + 2\zeta_6 x_3 x_6 + \zeta_6 x_6^2$

Maps:  $\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{12}^{-1} & -\zeta_{12} & 0 & 0 & 0 & 0 \\ 0 & -\zeta_{12} & -\zeta_{12}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6 & \zeta_6 & 0 \\ 0 & 0 & 0 & -\zeta_3 & 0 & 0 & \zeta_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\zeta_3 \\ 0 & 0 & 0 & 0 & 0 & \zeta_6 & 0 \end{bmatrix},$

$$\begin{bmatrix} \zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \zeta_3 & 0 & 0 & 0 & 0 \\ 0 & \zeta_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\zeta_3 & -\zeta_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Genus 7, Locus 3: Group (64, 41), signature (2,3,16), tetragonal

Ideal:  $x_0^2 + x_1x_2,$   
 $x_3x_4 - \zeta_8x_5x_6,$   
 $x_1^2 + x_3x_6 + ix_4x_5,$   
 $x_2^2 + ix_3x_6 + x_4x_5,$   
 $x_0x_1 + \zeta_{16}^7x_3^2 - \zeta_{16}^5x_5^2,$   
 $x_0x_2 - \zeta_{16}^7x_4^2 + \zeta_{16}^5x_6^2,$   
 $x_0x_3 - \zeta_{16}x_2x_6,$   
 $x_0x_4 + \zeta_{16}x_1x_5,$   
 $x_0x_5 + \zeta_{16}^7x_2x_4,$   
 $x_0x_6 - \zeta_{16}^7x_1x_3$

Maps:  $(x_0, \dots, x_6) \mapsto (-x_0, x_2, x_1, x_4, x_3, x_6, x_5),$   
 $(x_0, \dots, x_6) \mapsto (ix_0, -\zeta_8^3x_2, -\zeta_8x_1, x_6, -\zeta_8^3x_5, -\zeta_8x_4, x_3)$

Genus 7, Locus 4: Group (64, 38), signature (2,4,16), hyperelliptic  
 $y^2 = x^{16} - 1$

Genus 7, Locus 5: Group (56, 4), signature (2,4,28), hyperelliptic  
 $y^2 = x^{15} - x$

Genus 7, Locus 6: Group (54, 6), signature (2,6,9)

Ideal:  $x_1x_6 + x_2x_4 + x_3x_5,$   
 $x_0^2 - x_1x_6 + \zeta_6x_2x_4 - \zeta_3x_3x_5,$   
 $x_1x_4 + \zeta_3x_2x_5 - \zeta_6x_3x_6,$   
 $x_1x_5 + \zeta_3x_2x_6 - \zeta_6x_3x_4,$   
 $x_0x_1 - \zeta_6x_5^2 - x_4x_6,$   
 $x_0x_2 + x_6^2 - \zeta_3x_4x_5,$   
 $x_0x_3 + \zeta_3x_4^2 + \zeta_6x_5x_6,$   
 $x_0x_4 - x_1^2 + \zeta_3x_2x_3,$   
 $x_0x_5 - \zeta_3x_2^2 - \zeta_6x_1x_3,$   
 $x_0x_6 + \zeta_6x_3^2 + x_1x_2$

Maps:  $(x_0, \dots, x_6) \mapsto (-x_0, \zeta_9^5x_6, \zeta_9^8x_4, \zeta_9^2x_5, \zeta_9x_2, \zeta_9^7x_3, \zeta_9^4x_1),$   
 $(x_0, \dots, x_6) \mapsto (\zeta_6x_0, \zeta_3^2x_4, \zeta_3^2x_5, \zeta_3^2x_6, \zeta_3x_3, \zeta_3x_1, \zeta_3x_2)$

Genus 7, Locus 7: Group (54, 6), signature (2,6,9)

Complex conjugate of the previous curve

Genus 7, Locus 8: Group (54, 3), signature (2,6,9) cyclic trigonal

Trigonal equation:  $y^3 = x^9 - 1$

Ideal:  $2 \times 2$  minors of  $\begin{bmatrix} x_0 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_3 & x_4 & x_5 & x_6 \end{bmatrix},$  and

$$\begin{aligned} &x_0^3 - x_6^2 + x_2^3, \\ &x_0^2x_1 - x_6^2x_4 + x_2^2x_3, \\ &x_0x_1^2 - x_6^2x_5 + x_2^2x_4, \\ &x_1^3 - x_6^3 + x_2^2x_5 \end{aligned}$$

Maps:  $(x_0, \dots, x_6) \mapsto (x_1, x_0, -x_6, -x_5, -x_4, -x_3, -x_2),$   
 $(x_0, \dots, x_6) \mapsto (\zeta_9x_1, \zeta_9^2x_0, -\zeta_9x_6, -\zeta_9^2x_5, -\zeta_9^3x_4, -\zeta_9^4x_3, -\zeta_9^5x_2)$

Genus 7, Locus 9: Group (48, 32), signature (3,4,6). Has  $g_6^2$

Ideal:

$$\begin{aligned} & x_0^2 + x_3x_5 + \zeta_6x_3x_6 - \zeta_3x_4x_6, \\ & \sqrt{3}i(x_1x_3 - x_2x_4 + x_1x_5) - x_1x_6 + x_2x_5 - x_2x_6, \\ & \sqrt{3}i(2x_1x_4 - x_2x_5 + x_2x_6) + x_1x_5 - x_1x_6, \\ & \sqrt{3}i(2x_2x_3 + x_1x_6 - x_2x_5) + 3x_1x_5 + x_2x_6, \\ & -3(x_1^2 + x_3x_5 - x_3x_6) + \sqrt{3}i(x_4x_5 - x_4x_6) + 2x_5^2 + 2(\zeta_6 - 1)x_5x_6 - 2\zeta_6x_6^2, \\ & -3(x_1x_2 - x_4x_5) + \sqrt{3}i(x_3x_5 - x_3x_6 + x_4x_6 - x_5^2) + 2\zeta_6x_5x_6 - x_6^2, \\ & -3(x_2^2 - x_3x_5 - x_4x_6) + \sqrt{3}i(x_3x_6 - 3x_4x_5) + 2(\zeta_6 + 1)x_5x_6 + 2(\zeta_6 - 1)x_6^2, \\ & -3(2x_4^2 - x_3x_5 + x_3x_6) + \sqrt{3}i(-x_4x_5 + x_4x_6) + 2x_5^2 + 2(\zeta_6 - 1)x_5x_6 - 2\zeta_6x_6^2, \\ & -3(-2x_3x_4 + x_4x_5) + \sqrt{3}i(-x_3x_5 + x_3x_6 - x_4x_6 - x_5^2) + 2\zeta_6x_5x_6 - x_6^2, \\ & -3(2x_3^2 + x_3x_5 + x_4x_6) + \sqrt{3}i(-x_3x_6 + 3x_4x_5) + 2(\zeta_6 + 1)x_5x_6 + 2(\zeta_6 - 1)x_6^2 \end{aligned}$$

Maps:

$$\begin{aligned} (x_0, \dots, x_6) &\mapsto (\zeta_3x_0, -\zeta_6x_2, -\zeta_3x_1 - x_2, -x_3 + \zeta_3x_4, \zeta_6x_3, \zeta_3x_5, -\zeta_6x_5 + x_6), \\ (x_0, \dots, x_6) &\mapsto (-x_0, -x_2, x_1, x_4, -x - 3, \zeta_3x_5 - \zeta_6x_6, -\zeta_6x_5 - \zeta_3x_6) \end{aligned}$$

Genus 7, Locus 10: Group (42, 4), signature (2,6,21) cyclic trigonal

Trigonal equation:  $y^3 = x^8 - x$

Ideal:  $2 \times 2$  minors of  $\begin{bmatrix} x_0 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}$ , and

$$\begin{aligned} & x_0^3 - x_6^2x_2 + x_2^2x_3 \\ & x_0^2x_1 - x_6^2x_3 + x_2^2x_4, \\ & x_0x_1^2 - x_6^2x_4 + x_2^2x_5, \\ & x_1^3 - x_6^2x_5 + x_2^2x_6 \end{aligned}$$

Maps:

$$\begin{aligned} (x_0, \dots, x_6) &\mapsto (\zeta_7x_1, \zeta_7^6x_0, -\zeta_7^4x_6, -\zeta_7^2x_5, -x_4, -\zeta_7^5x_3, -\zeta_7^3x_2), \\ (x_0, \dots, x_6) &\mapsto (\zeta_{21}^{-2}x_1, \zeta_{21}^{-5}x_0, -\zeta_{21}^{-1}x_6, -\zeta_{21}^{-4}x_5, (\zeta_3 + 1)x_4, -\zeta_{21}^{11}x_3, -\zeta_{21}^8x_2) \end{aligned}$$

Genus 7, Group 11: Group (32, 11), signature (4,4,8). Has  $g_6^2$

Ideal:  $x_3x_5 + x_4x_6,$

$$\begin{aligned} & x_0^2 + x_1x_5 + ix_2x_6, \\ & x_1x_4 + ix_2x_3 + x_5x_6, \\ & x_1x_2 + x_3x_4, \\ & x_1x_6 + \zeta_8^3x_4x_5, \\ & x_2x_5 + \zeta_8x_3x_6, \\ & x_1^2 - ix_3^2 - \zeta_8^3x_5^2, \\ & x_2^2 + ix_4^2 + \zeta_8^3x_6^2, \\ & -ix_2x_4 + \zeta_8^3x_3^2, \\ & x_1x_3 - \zeta_8^3x_4^2 \end{aligned}$$

Maps:  $(x_0, \dots, x_6) \mapsto (-x_0, -x_2, x_1, -ix_4, -ix_3, ix_6, ix_5),$

$$(x_0, \dots, x_6) \mapsto (ix_0, x_1, ix_2, -x_3, -ix_4, -x_5, ix_6)$$

Genus 7, Locus 12: Group (32, 10), signature (4,4,8)

Ideal:  $x_1x_6 + \zeta_{16}^6x_2x_5 + x_3x_4,$

$$\begin{aligned} & x_1x_2 + x_5x_6, \\ & x_0^2 + x_1x_6 - \zeta_{16}^6x_2x_5, \\ & x_3x_6 - \zeta_{16}^4x_4x_5, \\ & x_1^2 - \zeta_{16}^7x_4^2 - \zeta_{16}^6x_5^2, \\ & x_2^2 + \zeta_{16}^3x_3^2 - \zeta_{16}^{10}x_6^2, \\ & -\zeta_{16}^2x_2x_6 + (\zeta_{16}^{16} + \zeta_{16}^8)x_4^2 - \zeta_{16}^7x_5^2, \\ & x_1x_5 + (-\zeta_{16}^{12} - \zeta_{16}^4)x_3^2 - \zeta_{16}^{11}x_6^2, \\ & x_1x_3 + \zeta_{16}^7x_4x_6, \\ & x_2x_4 + \zeta_{16}x_3x_5 \end{aligned}$$

Maps:  $(x_0, \dots, x_6) \mapsto (-x_0, -x_2, x_1, -\zeta_{16}^2x_4, -\zeta_{16}^6x_3, -\zeta_{16}^6x_6, -\zeta_{16}^2x_5),$

$$(x_0, \dots, x_6) \mapsto (ix_0, -\zeta_{16}^6x_2, -\zeta_{16}^2x_1, -ix_4, ix_3, -ix_6, -ix_5)$$

Genus 7, Locus 13: Group (30, 4), signature (2,15,30), hyperelliptic,  $y^2 = x^{15} - 1$

## REFERENCES

- [1] Jeffrey D. Achter and Rachel Pries, *The integral monodromy of hyperelliptic and trielliptic curves*, Math. Ann. **338** (2007), no. 1, 187–206, DOI 10.1007/s00208-006-0072-0. ↑9
- [2] Tatsuya Arakawa, *Automorphism groups of compact Riemann surfaces with invariant subsets*, Osaka J. Math. **37** (2000), no. 4, 823–846. ↑5
- [3] Thomas Breuer, *Characters and automorphism groups of compact Riemann surfaces*, London Mathematical Society Lecture Note Series, vol. 280, Cambridge University Press, Cambridge, 2000. ↑1, 2, 3, 5, 8
- [4] E. Bujalance, F. J. Cirre, and G. Gromadzki, *Groups of automorphisms of cyclic trigonal Riemann surfaces*, J. Algebra **322** (2009), no. 4, 1086–1103, DOI 10.1016/j.jalgebra.2009.05.017. ↑5
- [5] Antonio F. Costa and Milagros Izquierdo, *Maximal order of automorphisms of trigonal Riemann surfaces*, J. Algebra **323** (2010), no. 1, 27–31, DOI 10.1016/j.jalgebra.2009.09.041. ↑5
- [6] M. Conder, *Lists of regular maps*. <https://www.math.auckland.ac.nz/~conder/>. ↑1, 8
- [7] Vahid Dabbaghian-Abdoly, *An algorithm for constructing representations of finite groups*, J. Symbolic Comput. **39** (2005), no. 6, 671–688, DOI 10.1016/j.jsc.2005.01.002. ↑3
- [8] Vahid Dabbaghian and John D. Dixon, *Computing matrix representations*, Math. Comp. **79** (2010), no. 271, 1801–1810, DOI 10.1090/S0025-5718-10-02330-6. ↑3
- [9] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR1322960 (97a:13001) ↑5
- [10] H. M. Farkas and I. Kra, *Riemann surfaces*, 2nd ed., Graduate Texts in Mathematics, vol. 71, Springer-Verlag, New York, 1992. ↑3
- [11] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. A first course; Readings in Mathematics. ↑4
- [12] The GAP Group, *GAP: Groups, Algorithms, and Programming, a system for computational discrete algebra* (2015), available at <http://www.gap-system.org>. Version 4.7.8. ↑3
- [13] Naoki Inoue and Fumiharu Kato, *On the geometry of Wiman’s sextic*, J. Math. Kyoto Univ. **45** (2005), no. 4, 743–757. ↑12
- [14] Deepak Kapur, Yao Sun, and Dingkan Wang, *An efficient method for computing comprehensive Gröbner bases*, J. Symbolic Comput. **52** (2013), 124–142, DOI 10.1016/j.jsc.2012.05.015. ↑5
- [15] Izumi Kuribayashi and Akikazu Kuribayashi, *Automorphism groups of compact Riemann surfaces of genera three and four*, J. Pure Appl. Algebra **65** (1990), no. 3, 277–292, DOI 10.1016/0022-4049(90)90107-S. ↑8
- [16] Akikazu Kuribayashi and Hideyuki Kimura, *Automorphism groups of compact Riemann surfaces of genus five*, J. Algebra **134** (1990), no. 1, 80–103, DOI 10.1016/0021-8693(90)90212-7. ↑8
- [17] Dan Grayson and Mike Stillman, *Macaulay 2: a software system for research in algebraic geometry* (2015), available at <http://www.math.uiuc.edu/Macaulay2/>. Version 1.8. ↑6
- [18] A. M. Macbeath, *On a curve of genus 7*, Proc. London Math. Soc. (3) **15** (1965), 527–542. ↑14
- [19] K. Magaard, T. Shaska, S. Shpectorov, and H. Völklein, *The locus of curves with prescribed automorphism group*, Sūrikaisekikenkyūsho Kōkyūroku **1267** (2002), 112–141. Communications in arithmetic fundamental groups (Kyoto, 1999/2001). ↑1, 2, 8, 9, 11, 12, 14
- [20] School of Mathematics and Statistics Computational Algebra Research Group University of Sydney, *MAGMA computational algebra system* (2015), available at <http://magma.maths.usyd.edu.au/magma/>. Version 2.21-7. ↑1
- [21] David Mumford, *Lectures on curves on an algebraic surface*, With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966. ↑4
- [22] Katsusuke Nabeshima, *On the computation of parametric Gröbner bases for modules and syzygies*, Jpn. J. Ind. Appl. Math. **27** (2010), no. 2, 217–238, DOI 10.1007/s13160-010-0003-z. ↑5
- [23] Jennifer Paulhus, *Branching data for curves up to genus 48* (2015). <http://arxiv.org/abs/1512.07657>. ↑2, 8
- [24] Frank-Olaf Schreyer, *Syzygies of canonical curves and special linear series*, Math. Ann. **275** (1986), no. 1, 105–137, DOI 10.1007/BF01458587. ↑8, 14
- [25] Tanush Shaska, *Determining the automorphism group of a hyperelliptic curve*, Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2003, pp. 248–254 (electronic), DOI 10.1145/860854.860904, (to appear in print). ↑5, 9
- [26] Jan Stevens, *Deformations of singularities*, Lecture Notes in Mathematics, vol. 1811, Springer-Verlag, Berlin, 2003. ↑5
- [27] Manfred Streit, *Homology, Belyi functions and canonical curves*, Manuscripta Math. **90** (1996), no. 4, 489–509, DOI 10.1007/BF02568321. ↑4
- [28] D. Swinarski, *Supplementary files for "Equations of Riemann surfaces with large automorphism groups."* (2016). <http://faculty.fordham.edu/dswinarski/RiemannSurfaceAutomorphisms/>. ↑2, 4, 8, 9
- [29] Volker Weispfenning, *Comprehensive Gröbner bases*, J. Symbolic Comput. **14** (1992), no. 1, 1–29, DOI 10.1016/0747-7171(92)90023-W. ↑5
- [30] Anders Wiman, *Ueber die algebraischen Curven von den Geschlechtern  $p = 4, 5$ , und 6, welche eindeutige Transformationen in sich besitzen*, Bihang till K. Svenska Vet.-Akad. Handlingar **21** (1895), no. 3, 1–41 (German). ↑11, 12

DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, 113 W 60TH ST ROOM 813, NEW YORK, NY 10023  
 E-mail address: [dswinarski@fordham.edu](mailto:dswinarski@fordham.edu)